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LETTER TO THE EDITOR

Liouville condition, Nambu mechanics, and differential forms

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Abstract. Introducing a geometrical framework for description of Nambu mechanics, we give conditions to guarantee that an ordinary differential equation (ODE) may be written in the nambu form.

In Nambu mechanics [1], one considers dynamical systems, i.e. systems of time-autonomous ODEs, of the form

$$\dot{x}^i = f^i(x) = \epsilon_{j_1,\ldots,j_n} \delta^i_{j_1} \frac{\partial H_2}{\partial x^{j_2}} \cdots \frac{\partial H_n}{\partial x^{j_n}}. \tag{1}$$

Here $x \in M$ (a smooth manifold embedded in R^n), $f : M \to \mathrm{T}M$, and $H_j : M \to R$; ϵ is the completely antisymmetric (Levi–Civita) tensor on n indices. The functions $H_2, \ldots, H_n$ are the $n-1$ 'Nambu Hamiltonians' of the system.

It can immediately be checked that for such a system one has

$$\sum_{i=1}^{n} \frac{\partial f^i}{\partial x^i} = \mathrm{div}(f) = 0 \tag{2}$$

so that any system obeying Nambu mechanics (i.e. which can be written in the form (1) above) does automatically satisfy the Liouville condition (2). Indeed, the purpose of Nambu when proposing his generalization of Hamiltonian mechanics was to explore other forms of dynamics enforcing the Liouville condition, with the statistical mechanics setting in mind [1].

However, Nambu did not claim that systems of the form (1) are the only ones to satisfy the Liouville conditions (2), or equivalently that (2) implies that f can be written in the form (1). Such a statement was made in [2], and it has recently been observed [3] that this is in general false; i.e. there exist systems which cannot be written in form (1) but which do satisfy the Liouville condition (2).

The purpose of this letter is to point out that, once Nambu mechanics is expressed in natural geometrical terms, we can give geometrical conditions to guarantee that an ODE may be written in the 'Nambu form' (1), and to give such conditions.

Indeed, let us consider a generic dynamical system $\dot{x} = f(x)$ on $M \subseteq R^n$; the function f identifies a vector field $X : M \to \mathrm{T}M$, but also a dual $(n-1)$-form ω.

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The vector field is obviously given, in the x coordinates, by

$$X = f^i(x)\frac{\partial}{\partial x^i} \tag{3}$$

and the (dual) differential form is given by

$$\omega = \frac{1}{(n-1)!}\epsilon_{j_1,\ldots,j_n} f^{j_1}\,\mathrm{d}x^{j_2}\wedge\ldots\wedge\mathrm{d}x^{j_n}. \tag{4}$$

In geometrical terms, given the volume n-form $\Omega = \mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n$, the form ω is given by the interior product

$$\omega = i_X(\Omega). \tag{5}$$

We can enquire if ω is a closed form; to this aim, we just have to compute $\mathrm{d}\omega$, which yields

$$\mathrm{d}\omega = \left[\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}\right]\Omega. \tag{6}$$

Thus, we have shown how the Liouville condition reads on the dual form, i.e.:

Lemma 1. Let X be a vector field on $M\subseteq R^n$, given in local coordinates by $X = f^i(x)(\partial/\partial x^i)$, $f: M\to \mathbf{T}M$, and let ω be the $(n-1)$-form dual to X; let us denote by $\mathcal{L}_X$ the Lie derivative along X. Then, the following conditions are equivalent:

(i) $\mathrm{div}(f) = 0$;
(ii) $\mathrm{d}\omega = 0$;
(iii) $\mathcal{L}_X\Omega = 0$.

Then the problem of determining f which satisfies the Liouville condition is equivalent to the problem of determining the closed $(n-1)$-forms ω on M.

If we now use the expression (1) for f, we can compute ω explicitly:

$$\omega = \frac{1}{(n-1)!}\epsilon_{k_1,\ldots,k_n}\epsilon_{j_1,\ldots,j_n}\delta_{k_1 j_1}\left[\frac{\partial H_2}{\partial x^{j_2}}\,\mathrm{d}x^{j_2}\right]\wedge\ldots\wedge\left[\frac{\partial H_n}{\partial x^{j_n}}\,\mathrm{d}x^{j_n}\right] = \mathrm{d}H_2\wedge\ldots\wedge\mathrm{d}H_n. \tag{7}$$

It is clear that if ω is of form (7)—i.e. if f is in the form (1)—we have $\mathrm{d}\omega = 0$; but the converse is not true.

When a vector field (respectively, a differential form) is written in form (1) (respectively, in form (7)), we say it is in *Nambu form*.

Thus, we find that having f, or equivalently ω, in Nambu form is a sufficient, but not necessary, condition for X to satisfy the Liouville condition. Our geometric discussion confirms the findings of [3]; we hope it helps in clarifying the geometric structure underlying Nambu mechanics.

If we want to determine under which conditions the f satisfying (2) can also be written in form (1), we should ask under which condition the $(n-1)$-form ω is decomposable into $(n-1)$ closed one-forms (see also [4]).

We recall that a k-form μ is said to be *decomposable* if it may be written as the wedge product of k one-forms.

It is well known (see [5] p 22) that *every* $(n-1)$*-form over* M *is decomposable*, so that in particular ω is decomposable, i.e. we can write

$$\omega = \beta^2\wedge\ldots\wedge\beta^n \tag{8}$$

where the β^k are one-forms.

If we want to explicitly find the forms β^k, the idea is to build a local base $\{Y_1, \ldots, Y_n\}$ for the vector fields on M with $Y_1 = X$. Then, taking the dual base $\beta^1 \ldots \beta^n$ we can write ω in the form (8).

We would also like to point out that, using the theory of differential ideals [5], we can give an integrability condition for ω to be of form (8). In fact, let us consider the differential ideal $\mathcal{D}$ generated by the forms $\beta^2 \ldots \beta^n$. If $\mathcal{D}$ is closed with respect to the external differential, i.e. $\mathrm{d}\mathcal{D} \subset \mathcal{D}$, the Frobenius theorem [5] guarantees that, given $x_0 \in M$, there exists a local coordinate system $(\mathcal{U}, \psi^i)$ (with $i = 1 \ldots n$) around x_0 such that $\mathcal{D}$ is generated by $\mathrm{d}\psi^2 \ldots \mathrm{d}\psi^n$. Thus, we have:

Lemma 2. Let X be a vector field on $M \subseteq R^n$, and ω the dual $(n-1)$-form; let $\{Y_1, \ldots, Y_n\}$ be a (local) basis for vector fields on M, with $Y_1 = X$; let $\{\beta^1, \ldots, \beta^n\}$ is the dual (local) base of one-forms. If the differential ideal $\mathcal{D}$ generated by $\{\beta^2, \ldots, \beta^n\}$ is closed, then there is a (local) coordinate system $\{\psi^1, \ldots, \psi^n\}$ on M such that $\mathcal{D}$ is generated by $\{\mathrm{d}\psi^2, \ldots, \mathrm{d}\psi^n\}$, and ω may be written (locally) in the Nambu form $\omega = \mathrm{d}\beta^2 \wedge \ldots \wedge \mathrm{d}\beta^n$.

Notice that the ψ^k represent the (local) $(n-1)$ Nambu Hamiltonians.

It should be pointed out that we can equivalently express the closure condition of $\mathcal{D}$ in terms of vector fields; this amounts to asking that there exist a local base $\{Y_1, \ldots, Y_n\}$, with $Y_1 = X$, such that the Y_k's form a closed Lie algebra under the usual commutator. It is then possible to reformulate lemma 2 accordingly.

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References

[1] Nambu Y 1973 Generalized Hamiltonian dynamics *Phys. Rev.* D **7** 2405–12
[2] Ruggeri G J 1976 Comments on the Nambu mechanics *Lett. Nuovo Cimento* **17** 169–71
[3] Codriansky S, Navarro R and Pedroza M 1996 The Liouville condition and Nambu mechanics *J. Phys. A: Math. Gen.* **29** 1037–44
[4] Takhtajan L 1994 On foundation of the generalized Nambu mechanics *Comm. Math. Phys.* **160** 295–315
[5] Sternberg S 1964 *Lectures on Differential Geometry* (New York: Chelsea)